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Jeans collapse of turbulent gas clouds: tentative theory

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The linear Jeans stability problem in a turbulent medium is treated using a description of the large-scale motions, with the response of turbulence on the small scales being treated using a renormalization approach. This treatment shows how turbulence at scales smaller than the potentially collapsing scale builds up a *turbulent pressure* force which effectively resists compression, if the kinetic energy is sufficient to balance the gravitational attraction.

1. Introduction

This paper examines the conditions for gravitational instability in a statistically homogeneous turbulent fluid. The problem of stability of molecular clouds and the origin of protostellar density fluctuations is one motivation of this study.

Jeans (1902, 1929, pp. 345–347) assessed the condition for gravitational stability of an infinite homogeneous medium at rest, from the following dispersion equation:

$$\omega_k^2 - c_s^2 k^2 + 4\pi G \rho_0 = 0 \tag{1.1}$$

where k and ω_k are the wavenumber and frequency of a plane wave linear perturbation, c_s is the sound velocity, G the gravitational constant and ρ_0 the average mass density.

Fifty years later, Chandrasekhar (1951*b*) revisited the issue, and wrote (p. 27): '... it would follow from the enormous linear dimensions of the systems contemplated that the Reynolds number of the ensuing hydrodynamical motion will almost certainly be large enough for the medium to be considered highly turbulent. But turbulence is not a feature which is included in Jean's analysis. In view of the foregoing remarks it would seem worthwhile to recast Jeans's original arguments in the terminology of modern theories of turbulence.' And so he did; he wrote an equation of motion for the correlation function of density fluctuations under the stirring action of a given solenoidal turbulent velocity spectrum, in the limit of large separations; this equation admits spherical waves as a solution, leading to a dispersion equation identical to Jean's dispersion relation, except for c_s^2 which is replaced by $c_s^2 + \frac{1}{3}\langle v^2 \rangle$, where $\langle v^2 \rangle$ is the mean-square velocity of turbulence.

Over the past 20 years, radioastronomers have detected cold ($T < 30$ K) interstellar clouds. The classical analysis of Jeans applied to the stability of these clouds leads to the conclusion that the thermal gas pressure is insufficient to keep them in equilibrium. At the same time, the line widths observed show that these clouds are subject to macroscopic motions which carry an amount of kinetic energy much larger than the thermal internal energy of the gas, but comparable to the potential gravitational energy of the cloud. This coincidence has prompted people working in the field to admit that this *turbulent* velocity dispersion is counteracting the effect of gravity, keeping the clouds stable for periods much larger than a free fall time, in agreement with Chandrasekhar's predictions. But then it becomes somewhat puzzling that the gravitational instability leading to star formation would operate on scales less than the system's scale.

To answer this question, Bonazzola *et al.* (1987) discussed the idea that, as advocated by Chandrasekhar, a dispersion equation of the form (1.1) should hold in presence of turbulence, with a modified c_s^2 (*turbulent pressure*). That should take the following into account: a lump of matter of given size L, potentially subject to gravitational instability, can only feel as microturbulent agitation that part of the fluctuation spectrum which occurs at scales smaller than L. So, the stability criterion should not depend only on the total energy in the turbulence, but also on the slope of the turbulence spectrum. This leads to the possibility of small density enhancements becoming gravitationally unstable within larger units which remain stable. Their arguments were based on virial theorem considerations, and on the study of a two-dimensional numerical model.

This called for a more rigorous approach of the problem. In this paper, we study the Jeans instability problem in the limit of very long wavelengths ($k \rightarrow 0$), trying 'to recast Jeans's original arguments in the terminology of modern theories of turbulence', in Chandrasekhar's terms. The response of turbulence to the large-scale compression must then be calculated. To cope with that problem, we have chosen to describe their response in the framework of a 'renormalized' description of the larger-scale motions in the presence of smaller-scale ones. This technique has proved its ability to derive accurate results in other hydrodynamical problems (e.g. Yakhot & Orszag 1986*a*, *b*, 1987). Various versions of it were originally introduced by e.g. Forster, Nelson & Stephen (1977), Pouquet, Fournier & Sulem (1978), Moffatt (1983) for studies of passive scalar transport, incompressible turbulence and the dynamo mechanism in incompressible MHD. After this paper was submitted, Staroselsky *et al.* (1990) used these methods to study compressive turbulence in a gravitationless medium, and also reached the conclusion that the effective sound speed is scale-dependent.

Earlier approaches to the self-gravitating turbulent problem include the work of Sasao (1973) who improved upon Chandrasekhar (1951*b*) by a consistent application of the joint normal distribution hypothesis for the turbulent fields, leading to a source term for the density fluctuations which is not included in Chandrasekhar's treatment. Sasao obtained a system of integro-differential equations which were not solved. He only looked at one particular source term of density fluctuations, and ignored the other effects, which may be important for the stability study. His work shows how difficult the complete dynamical problem is. Our own view is also that Chandrasekhar's early approach is not entirely consistent. Our criticisms are to be found in Appendix A. Sasao's contribution, on the other hand highlights only one particular type of effect.

In this paper, we consider a somewhat different problem, i.e. the response to large-scale density perturbations imposed from outside, instead of trying to solve for the evolution of spontaneous density fluctuations. Chandrasekhar and Sasao considered the time evolution of the correlation of density fluctuations at all scales, while we consider the equation of motion of a linear perturbation at large scale of a steady-state turbulent flow. Our approach ignores other facets of the problem of a self-gravitating turbulent medium. These are briefly mentioned below.

First, the total angular momentum of our system is zero. A system with non-zero net angular momentum should be more stable. Second, a turbulent field at a scale comparable to the scale L at which we want to study the gravitational instability would constantly redistribute the matter in a larger volume and tend to break density fluctuations. If the fluid at higher density is dispersed before it has enough time to collapse (Léorat, Passot & Pouquet 1990), the large-scale turbulent velocity field has a stabilizing role. The breaking of density fluctuations into smaller units has been studied in detail in a different context by Higdon (1986).

Third, solenoidal turbulence generates density fluctuations (Lighthill 1952; Sasao 1973). This is certainly important to understand the nonlinear evolution of the density spectrum, but not for the problem of stability we consider. Fourth, the stability analysis is not complete if the *thermal* behaviour (for instance *adiabatic* or *isothermal* evolution) of the system is not known. The Jeans analysis says whether, in a uniform infinite medium at rest, a sine linear perturbation will grow or not under the action of gravity and pressure forces. But the nonlinear stability of the turbulent system with respect to an overall compression, for instance, depends on how $\partial P_t/\partial\rho$ varies with the density. This is a more difficult problem, which may be addressed in a future paper.

Our simpler problem, nevertheless, gives some insight into the nature and the effects of the effective turbulent pressure. In fact a result of our study is that the change of the turbulent velocity field responding to a local density perturbation affects the stability of this density perturbation. The turbulent velocity field at smaller scales actually responds to a compression as a pressure P_t, that is, with a force $-(\partial P_t/\partial\rho)\,\nabla\rho$ opposed to the gradient of the density. Certainly small-scale kinetic energy must show up as some form of pressure (volume density of energy), as is intuitive and appealing, when making energy balance arguments. However, the concept of a turbulent pressure force, in the sense defined just above is not so obviously meaningful, since there is no obvious reason why the average $\langle(\boldsymbol{v}\cdot\nabla)\,\boldsymbol{v}\rangle$ should yield a term proportional to $\langle v^2\rangle\,(\nabla\rho/\rho)$. Physically such a relation reflects the existence of some positive (or eventually null) correlation between the turbulent velocity field and the density field. Note that the absence of a turbulent pressure gradient force, in some limit cases would not necessarily conflict with the idea that an effective turbulent energy density shows up in the virial theorem.

We need a generalization of the concept of Jeans stability in a turbulent medium. We imagine a steady state of stationary turbulence with some low-wavenumber cut-off (K_c). We consider a density perturbation at scale $L > 2\pi/K_c$. It is enough for our purpose to calculate the changes caused in the turbulent fields to first order in this density perturbation. This idealization is somewhat unrealistic since probably the developed turbulent spectrum would extend up to the largest scales, but necessary for clearly decoupling the *collapse* problem from the more general and nonlinear problem of the establishment of the developed turbulent fluctuation spectrum in the self-gravitating compressible medium considered. Indeed, when the turbulent fluctuations on scale L become of high enough amplitude that the linear approximation is invalid for them, the very concept of Jeans stability completely breaks down. Therefore, for simplicity, we study an idealized model where the source of turbulence is on a scale smaller than that of the potentially collapsing unit under consideration, though we keep in mind that in the astrophysical conditions the question of gravitational collapse may just be an intrinsically nonlinear one. This is becoming more obvious in view of recent observational data on small-scale structure of molecular clouds (Falgarone & Pérault 1988; Falgarone, Phillips & Walker 1990). Nevertheless the idealized model may shed some light on the role of turbulence in controlling the evolution of subregions of gas clouds.

The dynamical renormalization group technique (RNG) was first introduced for the study of the dynamics of critical phenomena (Ma & Mazenko 1975). A number of authors, quoted above, applied this technique to specific problems of hydrodynamics. The approach to be followed here is similar to that of Yakhot & Orszag (1986*b*) in the sense that we do not apply any rescaling, in order to get the absolute value of renormalized transport coefficients.

In fact, a direct averaging on all scales smaller than any *a priori* given one is not possible when the associated Reynolds number is large, because the use of the first-order smoothing approximation (Bourret 1965) is then forbidden. A more appropriate way to achieve this global averaging is a recursive procedure of averaging on a series of narrow ranges of smaller scales. This recursive elimination of the smaller scales, whose effect becomes progressively absorbed in effective transport coefficients and, in the present case, in effective pressure, constitutes the essence of the renormalization technique. The invariance of the form of the new equations obtained when a new range of lengthscales is eliminated is a basic justification of this method. The equations obtained at each step are meant to be valid only for motions at a scale larger than those which have been already eliminated. This procedure can only be consistent if the renormalized Reynolds number remains small at each step for the smallest (remaining) scales. In practice the completion of this programme involves a number of other approximations which are difficult to justify, though they are usual practice both in hydrodynamics and in the field of critical phenomena (Moffatt 1983). These are described in Appendix B. It is still felt that the physics of eddy interactions remains at least qualitatively preserved with this procedure.

2. On the physical origin of the turbulent pressure force

In this section, we only want to discuss qualitatively the origin of the renormalization of the pressure force and viscosity. We consider a turbulent fluid subject to a linearizable perturbation at large scale. For simplicity, in this preliminary discussion, we ignore gravity. Since the fluid is compressible, it is convenient to use as a variable the momentum density, $\boldsymbol{\Phi} = \rho \boldsymbol{v}$, rather than the fluid velocity $\boldsymbol{v}$. The equations for the fluid motion are

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot \boldsymbol{\Phi}, \tag{2.1}$$

$$\frac{\partial \boldsymbol{\Phi}}{\partial t} = -\nabla \cdot \frac{\boldsymbol{\Phi}\boldsymbol{\Phi}}{\rho} - c_s^2 \nabla \rho + (\eta \nabla \cdot \nabla + (\tfrac{1}{3}\eta + \zeta)\nabla\nabla \cdot)\frac{\boldsymbol{\Phi}}{\rho}, \tag{2.2}$$

where ρ is the fluid mass density, $c_s^2 = \partial P/\partial \rho$ is the squared sound velocity, η and ζ are respectively the dynamic shear and bulk viscosities.

We consider a perturbation $(\tilde{\rho}, \tilde{\boldsymbol{\Phi}})$ around a stationary turbulent solution $(\rho, \boldsymbol{\Phi})$. It is assumed that the perturbation is induced at some large scale; the response of the turbulent field is examined.

We first want to show in this section, without detailed calculations, how the renormalization procedure gives rise to a 'turbulent viscosity' and a 'turbulent pressure'. The actual implementation of the renormalization programme is deferred to §3.

Let us split the momentum density into a large-scale (superscript L) and a small-scale part (superscript S), and assume that the density has only a large-scale part. Then the momentum equation can be written

$$\frac{\partial \Phi_i^{\mathrm{S}}}{\partial t}+\frac{\partial \Phi_i^{\mathrm{L}}}{\partial t}+\nabla_i c_{\mathrm{s}}^2 \rho^{\mathrm{L}} = -\nabla_j \frac{\Phi_i^{\mathrm{L}} \Phi_j^{\mathrm{L}}}{\rho^{\mathrm{L}}}-\nabla_j \frac{\Phi_i^{\mathrm{S}} \Phi_j^{\mathrm{L}}}{\rho^{\mathrm{L}}}-\nabla_j \frac{\Phi_i^{\mathrm{L}} \Phi_j^{\mathrm{S}}}{\rho^{\mathrm{L}}}-\nabla_j \frac{\Phi_i^{\mathrm{S}} \Phi_j^{\mathrm{S}}}{\rho^{\mathrm{L}}}$$
$$+\eta\Delta \frac{\Phi_i^{\mathrm{S}}}{\rho^{\mathrm{L}}}+\eta\Delta \frac{\Phi_i^{\mathrm{L}}}{\rho^{\mathrm{L}}}+(\tfrac{1}{3}\eta+\zeta)\nabla_i\nabla_j\left(\frac{\Phi_j^{\mathrm{S}}}{\rho^{\mathrm{L}}}+\frac{\Phi_j^{\mathrm{L}}}{\rho^{\mathrm{L}}}\right). \quad (2.3)$$

Separating from (2.3) the large-scale part we obtain

$$\frac{\partial \Phi_i^{\mathrm{L}}}{\partial t}+\nabla_i c_{\mathrm{s}}^2 \rho^{\mathrm{L}}+\nabla_j \frac{\phi_i^{\mathrm{L}} \Phi_j^{\mathrm{L}}}{\rho^{\mathrm{L}}}-\eta\Delta\frac{\Phi_i^{\mathrm{L}}}{\rho^{\mathrm{L}}}+(\tfrac{1}{3}\eta+\zeta)\nabla_i\nabla_j\frac{\Phi_j^{\mathrm{L}}}{\rho^{\mathrm{L}}} = -\nabla_j\frac{\langle \Phi_i^{\mathrm{S}}\Phi_j^{\mathrm{S}}\rangle}{\rho^{\mathrm{L}}}. \quad (2.4)$$

Expanding the divergence of the effective momentum flux we get

$$\frac{\partial \Phi_i^{\mathrm{L}}}{\partial t}+\nabla_i c_{\mathrm{s}}^2 \rho^{\mathrm{L}}+\nabla_j \frac{\Phi_i^{\mathrm{L}} \Phi_j^{\mathrm{L}}}{\rho^{\mathrm{L}}}-\eta\Delta\frac{\Phi_i^{\mathrm{L}}}{\rho^{\mathrm{L}}}+(\tfrac{1}{3}\eta+\zeta)\nabla_i\nabla_j\frac{\Phi_j^{\mathrm{L}}}{\rho^{\mathrm{L}}}$$
$$=+\frac{\langle \Phi_i^{\mathrm{S}}\Phi_j^{\mathrm{S}}\rangle}{(\rho^{\mathrm{L}})^2}\nabla_j\rho^{\mathrm{L}}-\frac{1}{\rho^{\mathrm{L}}}\nabla_j\langle \Phi_i^{\mathrm{S}}\Phi_j^{\mathrm{S}}\rangle. \quad (2.5)$$

The two terms on the right-hand side represent the effective pressure and viscosity forces. If the second term were treated considering $\langle \Phi_i^{\mathrm{S}}\Phi_j^{\mathrm{S}}\rangle$ as strictly homogeneous, it would vanish, leaving only the first term which would then appear, for diagonal $\langle \Phi_i^{\mathrm{S}}\Phi_j^{\mathrm{S}}\rangle$, as $-\nabla P_{\mathrm{eff}}$, P_{eff} being some effective pressure. Note that this pressure gradient would be opposite to $\nabla\rho$, a rather unusual situation, similar to a negative pressure! However the second term of (2.5) does not vanish, because $\langle \Phi_i^{\mathrm{S}}\Phi_j^{\mathrm{S}}\rangle$ is slightly inhomogeneous. We therefore have to calculate its gradient. To do so, we solve for the small-scale part ϕ^{S}, in the usual first-order smoothing approximation.

With the present assumptions, namely that the density is only large-scaled, this gives

$$\frac{\partial \Phi_i^{\mathrm{S}}}{\partial t}+(v_j^{\mathrm{L}}\nabla_j)\Phi_i^{\mathrm{S}} = -\Phi_i^{\mathrm{S}}(\nabla_j v_j^{\mathrm{L}})-\Phi_j^{\mathrm{S}}(\nabla_j v_i^{\mathrm{L}})+\frac{\eta}{\rho^{\mathrm{L}}}\Delta\Phi_i^{\mathrm{S}}+(\tfrac{1}{3}\eta+\zeta)\frac{1}{\rho^{\mathrm{L}}}\nabla_i\nabla_j\Phi_j^{\mathrm{S}}, \quad (2.6)$$

where we have reintroduced the velocity variable where appropriate. The operator on the left-hand side is the time-derivative following the large-scale motion. We can formally solve (2.6) by integrating along the unperturbed trajectories of this large-scale velocity field. This gives for $\Phi_j^{\mathrm{S}}(t)$, assuming it vanishes at $t=-\infty$,

$$\Phi_j^{\mathrm{S}}(t) = -\int_0^\infty \mathrm{d}\tau\, \Phi_j^{\mathrm{S}}(t-\tau)(\nabla_l v_l^{\mathrm{L}})(t-\tau)-\int_0^\infty \mathrm{d}\tau\, \Phi_l^{\mathrm{S}}(t-\tau)(\nabla_l v_j^{\mathrm{L}})(t-\tau)$$
$$+\Delta\int_0^\infty \mathrm{d}\tau\frac{\eta}{\rho^{\mathrm{L}}}\Phi_j^{\mathrm{S}}(t-\tau)+\int_0^\infty \mathrm{d}\tau\frac{\frac{1}{3}\eta+\zeta}{\rho^{\mathrm{L}}}\nabla_j\nabla_l\Phi_l^{\mathrm{S}}(t-\tau), \quad (2.7)$$

where the argument $(t-\tau)$ means that the associated quantity is to be taken at a time $(t-\tau)$ and at a spatial position $\boldsymbol{r}(t-\tau)$ which, following the large-scale motion, will be brought to $\boldsymbol{r}$ at time t. Since this solution will be used to calculate a correlation tensor, only times shorter than the coherence time of small-scale motions, τ_{coh}, are of interest. Similarly, $|\boldsymbol{r}(t-\tau)-\boldsymbol{r}(t)|$ is meant to be shorter than their correlation length,

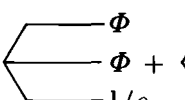
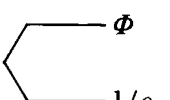
l_{coh}. Assuming these correlation lengths and times to be much smaller than those (L and T) characterizing the variations of the large-scale quantities, the preceding equation can be made Markovian by substituting t for $(t-\tau)$ in any quantity related to the large-scale dynamics, as is usual in first-order smoothing. In particular the large-scale Lagrangian motion reduces in this approximation to a constant-velocity one, and $\boldsymbol{r}(t-\tau) \approx \boldsymbol{r}-\boldsymbol{v}_{\text{L}}\tau$. Doing this, we obtain from (2.7) an approximate expression of the momentum correlation tensor in terms of the large-scale flow gradient:

$$\nabla_j \langle \Phi_i^{\text{S}} \Phi_j^{\text{S}} \rangle = -\nabla_j \int_0^\infty \mathrm{d}\tau \, \langle \Phi_i^{\text{S}}(t)\, \Phi_j^{\text{S}}(t-\tau)\rangle (\nabla_l v_l^{\text{L}}) - \nabla_j \int_0^\infty \mathrm{d}\tau \, \langle \Phi_i^{\text{S}}(t)\, \Phi_l^{\text{S}}(t-\tau)\rangle (\nabla_l v_j^{\text{L}})$$
$$+\nabla_j \frac{\eta}{\rho_{\text{L}}} \int_0^\infty \mathrm{d}\tau \, \langle \Phi_i^{\text{S}}(t)\, \Delta \Phi_j^{\text{S}}(t-\tau)\rangle + \nabla_j \frac{\frac{1}{3}\eta+\zeta}{\rho^{\text{L}}} \int_0^\infty \mathrm{d}\tau \, \langle \Phi_i^{\text{S}}(t)\, \nabla_j \nabla_l \Phi_l^{\text{S}}(t-\tau)\rangle. \tag{2.8}$$

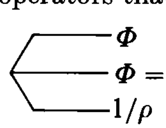
The averages involved are all isotropic, and can thus be simplified in an obvious way. The first two terms on the right-hand side of (2.8) renormalize the dissipative term and are contributions to an effective viscosity. The third term is a gradient of a quantity depending on the large-scale density variations and will eventually renormalize the pressure term, when combined with the first term on the right-hand side of (2.5). In a general way, the latter equation shows that the calculation of the large-scale dynamics involves that of the response of the turbulent field to the large-scale density perturbations. Equation (2.8) which is only a first-order expansion in (l_{coh}/L) and (τ_{coh}/T), shows how this response can be calculated.

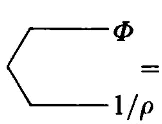
This programme will be completely carried out in §3, in a linear approximation. Since the corresponding equations become somewhat cumbersome, a diagrammatic representation has been adopted, which we describe now, with our linearization procedure, still retaining the simplified context adopted in this section (no gravity, no small-scale density fluctuations).

Let us write (2.2) symbolically as

$$\frac{\partial \boldsymbol{\Phi}}{\partial t} + c_{\text{s}}^2 \nabla \rho =$$  $+$  (2.9)

where the two- and three-branch vertices of equations (2.9) are symbolic representations of the operators that appear in (2.2), namely

 $= -\nabla \cdot \dfrac{\boldsymbol{\Phi}\boldsymbol{\Phi}}{\rho}$ (2.10)

 $= \eta \nabla^2 \dfrac{\boldsymbol{\Phi}}{\rho} + (\zeta + \frac{1}{3}\eta \nabla\nabla .)\dfrac{\boldsymbol{\Phi}}{\rho}$ (2.11)

Such a symbolic representation, which is not unusual in fluid mechanics, proves useful in coping with the general structure of equations, and makes it easier to recognize which terms contribute to which turbulent effect. The reader who feels better with equations written in explicit form can restore the regular algebraic writing by using the definitions (2.10) and (2.11).

We shall perform a double separation on the fields ρ and $\boldsymbol{\Phi}$. First we separate the 'background' from the 'perturbation', by writing

$$\rho(\boldsymbol{r},t) = \rho_0(\boldsymbol{r},t)+\tilde{\rho}(\boldsymbol{r},t), \tag{2.12}$$

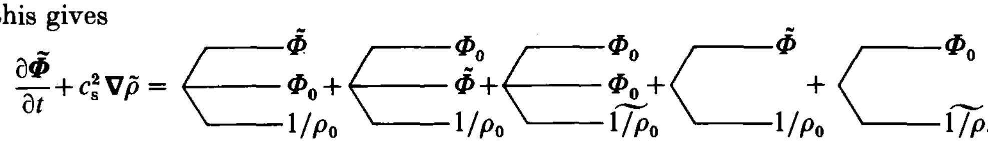
$$\boldsymbol{\Phi}(\boldsymbol{r},t) = \boldsymbol{\Phi}_0(\boldsymbol{r},t)+\tilde{\boldsymbol{\Phi}}(\boldsymbol{r},t). \tag{2.13}$$

Then, in each of those we separate a small-scale from a large-scale part. Such a decomposition can be used on a Fourier expansion. The large-scale part is that part which corresponds to Fourier vectors $\boldsymbol{k}$ such that $|\boldsymbol{k}|$ be smaller than some value. Taking the large-scale part of a field (resp. its small-scale part) defines a projection operator $\mathscr{L}$ (resp. $\mathscr{S}$). As before, the large-scale and small-scale parts will be denoted respectively by a superscript L and S. Note that L corresponds to small Fourier vectors and vice versa. For example

$$\frac{1}{\rho} = \frac{1}{\rho_0}+\left(\widetilde{\frac{1}{\rho}}\right) = \left(\frac{1}{\rho_0}\right)^{\mathrm{L}}+\left(\frac{1}{\rho_0}\right)^{\mathrm{S}}+\left(\widetilde{\frac{1}{\rho}}\right)^{\mathrm{L}}+\left(\widetilde{\frac{1}{\rho}}\right)^{\mathrm{S}}, \tag{2.14}$$

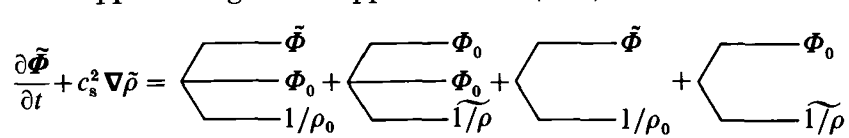
$$\boldsymbol{\Phi} = \boldsymbol{\Phi}_0+\tilde{\boldsymbol{\Phi}} = \boldsymbol{\Phi}_0^{\mathrm{L}}+\boldsymbol{\Phi}_0^{\mathrm{S}}+\tilde{\boldsymbol{\Phi}}^{\mathrm{L}}+\tilde{\boldsymbol{\Phi}}^{\mathrm{S}}. \tag{2.15}$$

We seek an equation for the large-scale part of the perturbation, averaging on the realizations of the small-scale part of the background and perturbation fields. The perturbation part of (2.2) is obtained by inserting (2.12) and (2.13) in (2.9), and subtracting the unperturbed form of this equation. Linearizing for the perturbation, this gives



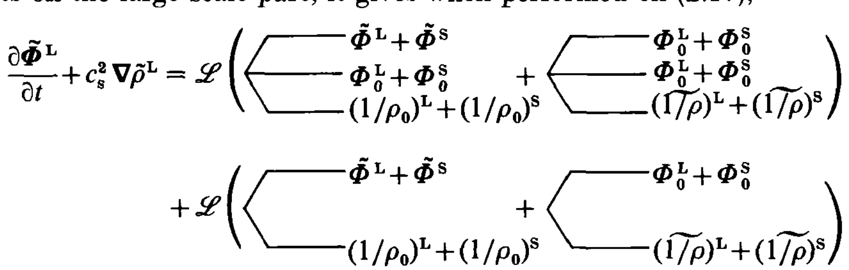
(2.16)

For brevity, similar graphs resulting from non-symmetric vertices will be represented by only one graph. This applies to the first two terms on the right-hand side of the above equation. Numerical coefficients will be omitted in the graphic equations in this section, since we emphasize only qualitative aspects of the procedure. However, all relevant integrals are taken into account with their proper weight in the application given in Appendix B. So (2.16) can be written more simply as



(2.17)

The functions in (2.17), in particular those on which vertices operators act, can be separated into their small- and large-scale components. $\mathscr{L}$ being the operator which projects on the large-scale part, it gives when performed on (2.17),



(2.18)

Each term on the right-hand side of (2.18) can be decomposed into its large- and small-scale part in the usual way. Due attention should be paid to the fact that the product of two small-scale fields has a large-scale part, which, in terms of Fourier analysis, means that the sum of two large-wave vectors may be small. We ignore effects due to compressibility on small scales, both in fluctuations and in the background turbulence. The effects are taken into account in the complete computation of the next section; the presentation adopted here is only meant to explain how turbulence responds to perturbations at larger scale in a simplified situation. We also ignore 'near-grid' couplings, by which we mean that the sum of two small wave vectors is considered as always small, although the definition given for operators $\mathscr{L}$ and $\mathscr{S}$ would, with full rigour, sometimes regard it as large (see Rose 1977). Neglecting, as mentioned earlier, $(1/\rho_0)^{\mathrm{S}}$, we obtain

$$\mathscr{L}\left\langle\begin{matrix}\Phi_0\\ \Phi_0\\ (\widetilde{1/\rho})\end{matrix}\right. = \left\langle\begin{matrix}\Phi_0^{\mathrm{L}}\\ \Phi_0^{\mathrm{L}}\\ (\widetilde{1/\rho})^{\mathrm{L}}\end{matrix}\right. + \mathscr{L}\left\langle\begin{matrix}\Phi_0^{\mathrm{S}}\\ \Phi_0^{\mathrm{S}}\\ (\widetilde{1/\rho})^{\mathrm{L}}\end{matrix}\right., \qquad (2.19)$$

$$\mathscr{L}\left\langle\begin{matrix}\tilde{\Phi}\\ \Phi_0\\ (1/\rho_0)\end{matrix}\right. = \left\langle\begin{matrix}\tilde{\Phi}^{\mathrm{L}}\\ \Phi_0^{\mathrm{L}}\\ (1/\rho_0)^{\mathrm{L}}\end{matrix}\right. + \mathscr{L}\left\langle\begin{matrix}\tilde{\Phi}^{\mathrm{S}}\\ \Phi_0^{\mathrm{S}}\\ (1/\rho_0)^{\mathrm{L}}\end{matrix}\right. \qquad (2.20)$$

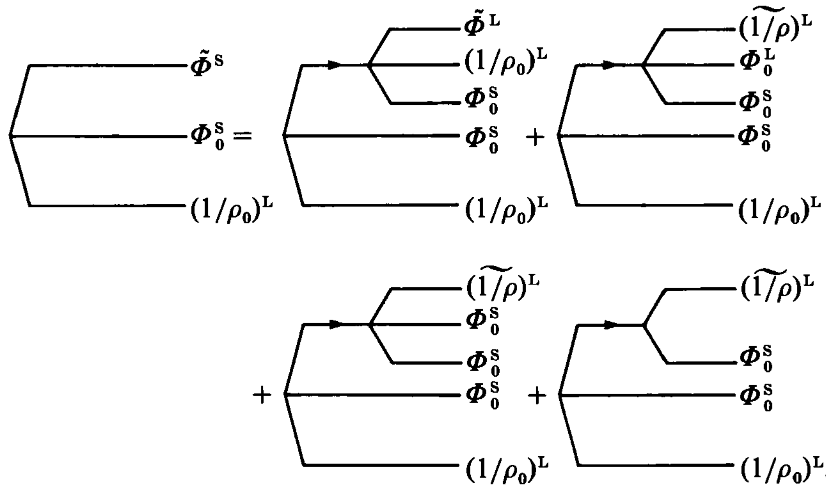
The projection operator $\mathscr{L}$ should remain in front of the second terms of the right-hand sides of (2.19) and (2.20), since the diagrams on which it acts also have a small-scale part. However, later on, this will be omitted from the notation, it being obvious that a large-scale equation cannot contain a small-scale part and conversely. The small-scale part of (2.17) is obtained in the same way; with the same assumptions and conventions, it can be written

$$\frac{\partial\tilde{\Phi}^{\mathrm{S}}}{\partial t} + c_{\mathrm{s}}^2\,\nabla\tilde{\rho}^{\mathrm{S}} = \left\langle\begin{matrix}\tilde{\Phi}^{\mathrm{S}}\\ \Phi_0^{\mathrm{S}}\\ (1/\rho_0)^{\mathrm{L}}\end{matrix}\right. + \left\langle\begin{matrix}\Phi_0^{\mathrm{S}}\\ \Phi_0^{\mathrm{L}}\\ (\widetilde{1/\rho})^{\mathrm{L}}\end{matrix}\right. + \left\langle\begin{matrix}\Phi_0^{\mathrm{S}}\\ \Phi_0^{\mathrm{S}}\\ (\widetilde{1/\rho})^{\mathrm{L}}\end{matrix}\right. + \left\langle\begin{matrix}\Phi_0^{\mathrm{S}}\\ \\ (\widetilde{1/\rho})^{\mathrm{L}}\end{matrix}\right. \qquad (2.21)$$

This can be solved, also using (2.1), for $\tilde{\Phi}^{\mathrm{S}}$ in terms of the right-hand side of (2.21). Let us represent the propagator, i.e. the linear operator which solves the linearized system (2.1)–(2.2), or equivalently here (2.1) and (2.21), by an arrow. Then we can write

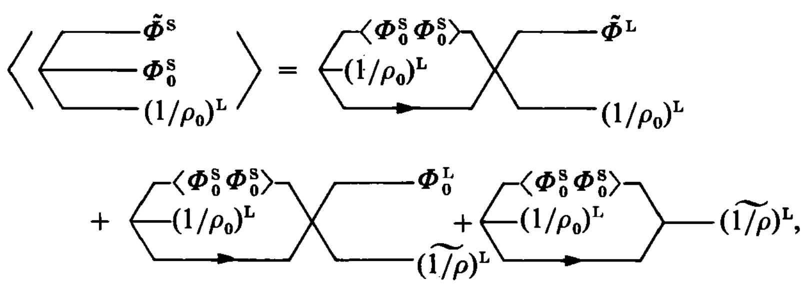
$$\tilde{\Phi}^{\mathrm{S}} = \rightarrow\left\langle\begin{matrix}\tilde{\Phi}^{\mathrm{S}}\\ \Phi_0^{\mathrm{L}}\\ (1/\rho_0)^{\mathrm{L}}\end{matrix}\right. + \rightarrow\left\langle\begin{matrix}\Phi_0^{\mathrm{S}}\\ \Phi_0^{\mathrm{L}}\\ (\widetilde{1/\rho})^{\mathrm{L}}\end{matrix}\right. + \rightarrow\left\langle\begin{matrix}\Phi_0^{\mathrm{S}}\\ \Phi_0^{\mathrm{S}}\\ (\widetilde{1/\rho})^{\mathrm{L}}\end{matrix}\right. + \rightarrow\left\langle\begin{matrix}\Phi_0^{\mathrm{S}}\\ \\ (\widetilde{1/\rho})^{\mathrm{L}}\end{matrix}\right. \qquad (2.22)$$

Now we substitute this solution into the second term on the right-hand side of (2.20), thus obtaining for the large-scale part of the three-branch diagram:



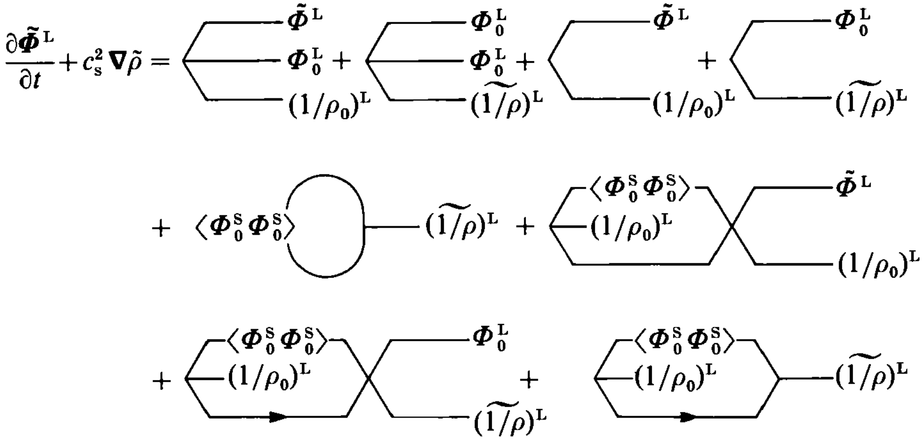
(2.23)

The perturbed equation for the large-scale part of the flow, (2.18), can now be obtained by selectively averaging (2.23) on the small scales, and similar expressions in (2.19) and (2.20) as well. In the spirit of the first-order smoothing approximation, which is meant to be 'effectively' valid at each scale, because the effective Reynolds number is assumed to remain of order unity at each scale reached by the iterative averaging process, we neglect the third-order moments (third term on the right-hand side of (2.23). Such a procedure replaces (2.23) by the following:



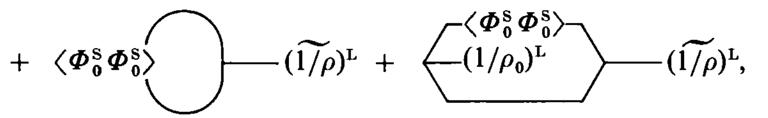
(2.24)

where the graphical notation has been adapted by allowing the branches associated with the arguments which enter in each vertex to emerge from the vertices in any direction; the quantities which are subject to an averaging procedure, denoted by angle brackets, have been brought next to each other, causing loops to appear in the diagrams. Proceeding in this same way for all the terms which appear in (2.18) the latter can be written:



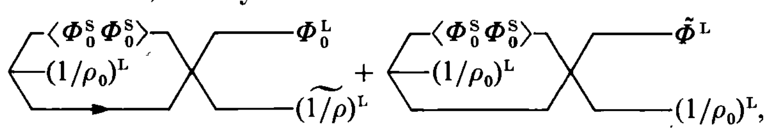
(2.25)

Those diagrams on the right-hand side of (2.25) which involve small-scale averages $\langle \phi_0^{\mathrm{S}} \phi_0^{\mathrm{S}} \rangle$ fall into two groups. Two terms turn out to be directly proportional to $\boldsymbol{\nabla}\tilde{\rho}$, namely



(2.26)

while the other two, namely



(2.27)

are obviously the linear approximation to a term which renormalizes the dissipative term of (2.2), because they turn out to act on the product $(\boldsymbol{\Phi}/\rho)$ in the same way as (2.11). The turbulent viscosity is an average momentum transport effect. It is the result of the advective term of the equation of motion, and is similar in nature to the effect of renormalized transport by turbulent media, well known in the theory of passive transport (e.g. Moffatt 1981) or to the renormalized viscosity associated with incompressible turbulence (Forster *et al.* 1977).

Equation (2.25) is the linearized form of both (2.9) and (2.5). The four first diagrams on its right-hand side are obviously the linearized form of the inertia and viscosity terms on the left-hand side of (2.5). The diagram which involves no propagator represents the first term of the right-hand side of (2.5). The second term on the right-hand side of (2.5) is made explicit in (2.8), the last term of which is just the second diagram of (2.26), while the other two constitute an effective viscosity term, represented by a second-order operator acting on $\boldsymbol{v}\,(=\boldsymbol{\Phi}/\rho)$. The two diagrams of (2.27) are actually the linear approximation to this term.

Iteration of the elementary step just described, down to a wavenumber k, gives the turbulent viscosity and pressure entering the momentum density equation for wave numbers smaller than k. This programme is completed for the Jeans problem in the next section, this time ignoring no relevant terms. Details are given in Appendix B.

3. Explicit renormalization of the Jeans problem

3.1. *The Jeans problem in a turbulent medium*

The hydrodynamical equations of a self-gravitating isothermal fluid are

$$\frac{\partial \rho}{\partial t} = -\nabla_j \Phi_j, \tag{3.1}$$

$$\frac{\partial \Phi_i}{\partial t} = -\nabla_j \frac{\Phi_i \Phi_j}{\rho} - \nabla_i \rho c_s^2 + \eta \nabla_j \nabla_j \frac{\Phi_i}{\rho} + (\tfrac{1}{3}\eta + \zeta) \nabla_i \nabla_j \frac{\Phi_j}{\rho} + \rho g_i + F_i, \tag{3.2}$$

$$\nabla_i g_i = -4\pi G\rho, \tag{3.3}$$

where $\boldsymbol{g}$ is the gravitational acceleration, and $\boldsymbol{F}$ some external source of agitation, acting at large scale, other terms are defined in §2. Initial and boundary conditions, as well as properties of the external force field stirring the fluid are not considered here. We only assume that we have a stationary turbulent solution of these equations. The turbulence is assumed to be homogeneous and isotropic. The usual convention of summation over the three space components for repeated indices is applied throughout the paper.

If the nonlinear terms are neglected we can obtain the dispersion equation for the density fluctuation $\rho_1 \mathrm{e}^{-\mathrm{i}\omega_k t + \mathrm{i}\boldsymbol{k}\cdot\boldsymbol{x}}$ around a hypothetical static equilibrium configuration of uniform density ρ_0 extending to infinity (although this is not justified unless the medium is an expanding universe, as in Jeans' 1902 paper):

$$\omega_k^2 - c_s^2 k^2 + \mathrm{i}\omega_k k^2 \frac{1}{\rho_0}(\tfrac{4}{3}\eta + \zeta) + 4\pi G\rho_0 = 0. \tag{3.4}$$

From this equation it is seen, as is well known, that the equilibrium is unstable if the wavenumber modulus k of the perturbation is less than the Jeans critical wavenumber k_J:

$$k_J^2 = \Omega_J^2 / c_s^2, \tag{3.5}$$

where Ω_J is the Jeans frequency, related to the average density, ρ_0, by

$$\Omega_J = (4\pi G\rho_0)^{\frac{1}{2}}. \tag{3.6}$$

As explained in the introduction we may ask what happens in the presence of turbulence. When discussing a stability problem we have to define a reference state around which the evolution of the perturbations can be studied in the presence of turbulence; we imagine a steady state with stationary turbulence.

Such stationary states must be maintained by some sort of forcing, which need not be described in any detail. We simply assume that a random force field covers a range of spatial scales from some wavenumber K_c to infinity, and we assume that K_c is much larger than the wavenumber k of the perturbation of interest. For such an external force spectrum, the velocity field spectrum develops below the cutoff, K_c, a low-wavenumber spectrum in k^4, which falls off quite rapidly when the wavelength grows larger (J. Léorat, private communication). As a result, the dominant contribution to the motions at wavenumber k can indeed be attributed to the perturbation that we are going to study, rather than to the low-frequency tail of the stationary turbulence spectrum. The Jeans number, K_J, is of course also assumed to be smaller than K_c, as is also the wavenumber of interest, k. This is represented in figure 1.

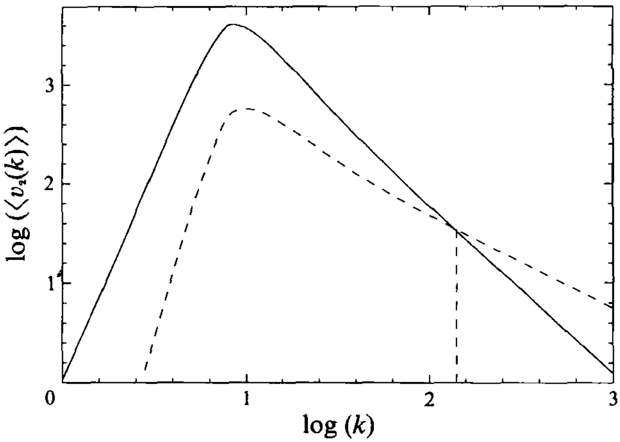


FIGURE 1. A qualitative sketch of typical turbulence spectra for the compressive part (dashed line) and solenoidal part (full line) of the velocity field. This latter part peaks at a wavenumber where turbulence is supposedly excited. It extends towards larger wavenumbers as a $k^{-\frac{5}{3}}$-spectrum and towards lower ones as a k^4-spectrum. The compressive part has a slope smaller than the incompressible one at large wavenumbers (Léorat *et al.* 1990) and dominates at $k > K_x$ (vertical dashed line). It is dominated at small k by the incompressible part (Staroselsky *et al.* 1990).

To perform explicit calculations we also need to introduce some assumption concerning the frequency spectrum of velocity fluctuations. For a given wavenumber p, we assume that the temporal frequency spectrum is a function which peaks at some low frequency, Ω_c, (which could be zero). This frequency is assumed to be independent of p. The spectrum then takes significant values only in a range of frequencies lower than the corresponding 'effective' sound wave frequency relative to wavenumber p. Under this assumption, the exact shape of this temporal frequency spectrum is irrelevant. Hence, the assumed independence of Ω_c of p is not expected to yield singular results either. In fact the general form of the equations obtained in §§4.1 and 4.2 is independent of this particular assumption, different dependence of Ω_c on p would alter some of the numerical coefficients which appear in equations like (4.2), (4.5) or (4.8). Physically this means that the temporal behaviour at a given spatial wavenumber is dominated by advection, with a velocity smaller than the effective sound speed, rather than by sound propagation effects. Then Ω_c and K_c should be related by a relation like

$$\Omega_c \approx V_c K_c, \tag{3.7}$$

where V_c is a typical advection velocity given, in terms of the space–time power spectrum of the velocity field, by an expression like

$$V_c^2 \approx K_c \int_{-\infty}^{+\infty} V^2(K_c, \omega)\, d\omega. \tag{3.8}$$

The Jeans frequency $\Omega_J = c_s K_J$ is much smaller than Ω_c. We shall consider the response of the turbulence to perturbations at small frequency ω, smaller than both Ω_c and Ω_J.

It is convenient to define a variable $\boldsymbol{\phi}$ having dimension of velocity by dividing the momentum density $\boldsymbol{\Phi}$ by the average density of the system ρ_0, $\boldsymbol{\phi} = \boldsymbol{\Phi}/\rho_0$. Fourier transforms, denoted by a hat, are defined as, for example,

$$\phi_i(x,t) = \iiiint_{-\infty}^{+\infty} d^3k\, d\omega_k\, e^{-i(\omega_k t - k\cdot x)} \hat{\phi}_i(k, \omega_k). \tag{3.9}$$

For compactness we introduce Fourier 4-vectors:

$$\boldsymbol{k} = (k_x, k_y, k_z, \omega), \tag{3.10}$$

$$\mathrm{d}\boldsymbol{k} = \mathrm{d}^3k\,\mathrm{d}\omega. \tag{3.11}$$

When special mention is made of frequency associated to 4-wave vector $\boldsymbol{k}$, the frequency will be denoted $\omega_{\boldsymbol{k}}$. But this does not imply any kind of dispersion relation. We define the transverse and longitudinal components of the spectrum of the variable $\boldsymbol{\phi}$, $\Psi^{\perp}(p, \omega_p)$ and $\Psi^{\parallel}(p, \omega_p)$ as

$$4\pi p^2 \langle \hat{\phi}_i(p, \omega_p)\, \hat{\phi}_j(q, \omega_q)\rangle = \delta(p+q)\,\delta(\omega_p+\omega_q)\left(\Psi^{\perp}(p, \omega_p)\left(\delta_{ij} - \frac{p_i p_j}{p^2}\right) + \Psi^{\parallel}(p, \omega_p)\frac{p_i p_j}{p^2}\right). \tag{3.12}$$

Numerical simulations of supersonic turbulence (Passot, Pouquet & Woodward 1987) have shown that in dimension-2, the spectrum of the transverse component is steeper than the spectrum of the longitudinal component. We assume that the three-dimensional turbulent flow that we consider has the same property. K_x is defined such that compression dominates at wavenumbers larger than K_x. The assumption made in this paper of small density fluctuations implies that the integral over p and ω_p of $\Psi^{\parallel}(p, \omega_p)$ is smaller than the thermal energy c_s^2. As we also assume that the turbulence contains a significant amount of energy, the integral of $\Psi^{\perp}(p, \omega_p)$ will be larger than that of $\Psi^{\parallel}(p, \omega_p)$. It is thus reasonable to assume that $\Psi^{\perp}$ will dominate the spectrum for all wavenumbers except those larger than K_x which contain only a small fraction of the turbulent energy.

3.2. *Equations in Fourier space, and graphical representation*

In order to apply the renormalization procedure in the next section, the general equations (3.1)–(3.3) must be written in Fourier space. To get rid of the $1/\rho$ nonlinearities, the equations will be considered in the low-Mach-number regime where the density fluctuations $n\rho_0$ around the space-averaged value ρ_0 are small. To order n^2, the equations take the form

$$\partial n/\partial t = -\nabla_j \phi_j, \tag{3.13}$$

$$\frac{\partial \phi_i}{\partial t} = -\nabla_j \phi_i \phi_j (1-n+n^2) - \nabla_i n c_s^2 + \nu_0 \nabla_j \nabla_j \phi_i (1-n+n^2) + \mu_0 \nabla_i \nabla_j \phi_j (1-n+n^2) + (1+n)\, g_i + f_i, \tag{3.14}$$

$$\nabla_i g_i = -\Omega_J^2\, n, \tag{3.15}$$

where Ω_J is the Jeans frequency, as above, and the momentum density $\boldsymbol{\Phi}$ has been divided by ρ_0, as explained above, as have the force field F and the dynamic viscosities η and ζ. The new variables $\boldsymbol{\phi}, \boldsymbol{f}$, have the dimensions of a velocity and an acceleration respectively, and ν_0 and μ_0 have the dimension of kinematical viscosities:

$$\nu_0 = \eta/\rho_0 \tag{3.16}$$

$$\mu_0 = \eta/3\rho_0 + \zeta/\rho_0. \tag{3.17}$$

Although the bulk viscosity is assumed to vanish in the original equation, we keep a second independent viscosity coefficient μ_0 in order to allow for the growth of turbulent bulk viscosity in the renormalization procedure (§3.3).

The solution of the mass conservation equation in Fourier space is, simply,

$$\hat{n} = \hat{\phi}_j k_j/\omega_k. \tag{3.18}$$

Substituting $\hat{n}$ in the Poisson equation fields gives

$$\hat{g}_i = i\frac{\Omega_\mathrm{J}^2}{\omega_k}\frac{\hat{\phi}_j k_j}{k^2}k_i. \tag{3.19}$$

Substitution of $\hat{n}$ and $\hat{g}$ in the momentum equation (3.13) gives an equation where only $\hat{\phi}$ remains. Nonlinear terms of third order are kept for consistency in the further developments. Nonlinear terms of higher order are dropped. The momentum equation then becomes

$$\begin{aligned}\mathcal{G}_{ij}^{-1}(\boldsymbol{k})\,\hat{\phi}_j(\boldsymbol{k}) = \hat{f}_i(\boldsymbol{k}) + \lambda\int \mathrm{d}\boldsymbol{p}\,\mathrm{d}\boldsymbol{q}\,\delta(\boldsymbol{p}+\boldsymbol{q}-\boldsymbol{k})\,\mathcal{Q}_{ijl}(\boldsymbol{k},\boldsymbol{q},\boldsymbol{q})\,\hat{\phi}_j(\boldsymbol{p})\,\hat{\phi}_l(\boldsymbol{q})\\ +\lambda^2\int \mathrm{d}\boldsymbol{p}\,\mathrm{d}\boldsymbol{q}\,\mathrm{d}\boldsymbol{r}\,\delta(\boldsymbol{p}+\boldsymbol{q}+\boldsymbol{r}-\boldsymbol{k})\,\mathcal{S}_{ijlm}(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q},\boldsymbol{r})\,\hat{\phi}_j(\boldsymbol{p})\,\hat{\phi}_l(\boldsymbol{q})\,\hat{\phi}_m(\boldsymbol{r})\end{aligned} \tag{3.20}$$

with the definitions:

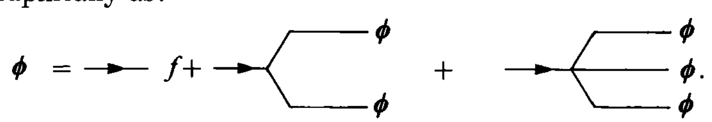
$$\mathcal{G}_{ij}^{-1}(\boldsymbol{k}) = (-\mathrm{i}\omega_k+\nu_0 k^2)\left(\delta_{ij}-\frac{k_i k_j}{k^2}\right)+\left(-\mathrm{i}\omega_k+\left(\mathrm{i}\frac{c_\mathrm{s}^2}{\omega_k}+\nu_0+\mu_0\right)k^2-\mathrm{i}\frac{\Omega_\mathrm{J}^2}{\omega_k}\right)\frac{k_i k_j}{k^2}, \tag{3.21}$$

$$\mathcal{Q}_{ijl}(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q}) = \mathrm{i}k_j\,\delta_{il}+\frac{q_l}{\omega_q}(\nu_0 k^2\delta_{ij}+\mu_0 k_i k_j)+\mathrm{i}\Omega_\mathrm{J}^2\frac{p_i p_j}{\omega_p p^2}\frac{q_l}{\omega_q}, \tag{3.22}$$

$$\mathcal{S}_{ijlm}(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q},\boldsymbol{r}) = \mathrm{i}k_j\,\delta_{ij}\frac{r_m}{\omega_r}-\nu_0 k^2\delta_{ij}\frac{q_l}{\omega_q}\frac{r_m}{\omega_r}-\mu_0 k_i k_j\frac{q_l}{\omega_q}\frac{r_m}{\omega_r}. \tag{3.23}$$

λ is a constant, equal to 1, explicitly written in order to keep track of the order in nonlinear couplings in the perturbation development.

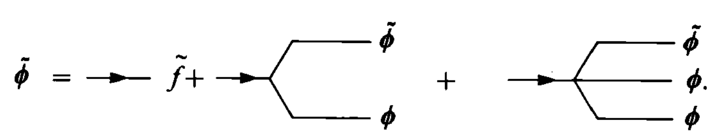
Representing now the propagator $\mathcal{G}$ with a line with arrow, the second-order nonlinear couplings with a vertex with two outgoing branches, and the third-order nonlinear couplings with a vertex with three outgoing branches, equation (3.14) can be written graphically as:



(3.24)

The wave vectors and frequencies must follow conservation laws at each vertex, i.e. the incoming one must equal the sum of the outgoing two or three.

Let us now consider a small perturbation to the stationary momentum and force fields. The perturbed quantities verify, in a linear approximation, the following equation



(3.25)

where the perturbation is represented with a tilda. The unperturbed momentum density field is considered here as known (at least statistically), and the original dynamical problem is transformed into a simpler kinematical problem of passive transport with several vertices.

3.3. *Application of the renormalization procedure*

The computation corresponding to one elementary step of the renormalization procedure is presented in detail in Appendix B, in practice for the first step. Starting from the perturbed equation (3.25) for wavenumbers in the range $k \leqslant \Lambda$, this step leads to similar equations for wavenumbers in the range $k \leqslant \Lambda\,\mathrm{e}^{-l}$, where the effects of turbulence in the range $N\mathrm{e}^{-l} < k \leqslant \Lambda$ (high wavenumbers), at lower order, appear as corrections to the coefficients c_s^2, ν_0 and μ_0 of the new equations. For clarity these renormalized coefficients will be written ν_t, μ_t and $c_t^2 = \partial P_t/\partial\rho$.

The corrections obtained for one step are listed in Appendix B. These corrections depend on frequency integrals of the spectrum of the momentum density $I_n^{\perp}(p)$ and $I_n^{\parallel}(p)$ defined as

$$I_n^{\perp}(p) = \int_{-\infty}^{\infty} \mathrm{d}\omega\,\omega^n\,\Psi^{\perp}(p,\omega), \tag{3.26}$$

$$I_n^{\parallel}(p) = \int_{-\infty}^{\infty} \mathrm{d}\omega\,\omega^n\,\Psi^{\parallel}(p,\omega), \tag{3.27}$$

where $\Psi^{\perp}$ and $\Omega^{\parallel}$ are the transverse and longitudinal components of the spectrum of the momentum density, already defined.

As long as the relation

$$\Omega_c^2 < \nu_t^2 p^4 \tag{3.28}$$

is valid, which is so under the assumptions previously made on the spectrum (see text before (3.7)), the leading correction terms are given by the expressions

$$\Delta c_t^2 = \int_{\Lambda\mathrm{e}^{-l}<p<\Lambda} \frac{2}{3} I_0^{\perp}\,\mathrm{d}p, \tag{3.29}$$

$$\Delta\nu_t = \int_{\Lambda\mathrm{e}^{-l}<p<\Lambda} \frac{8}{15}\frac{I_0^{\perp}}{\nu_t p^2}\,\mathrm{d}p, \tag{3.30}$$

$$\Delta\mu_t = \int_{\Lambda\mathrm{e}^{-l}<p<\Lambda} -\frac{4}{15}\frac{I_0^{\perp}}{\nu_t p^2}\,\mathrm{d}p, \tag{3.31}$$

$$\Delta\Omega_J^2 = 0. \tag{3.32}$$

The procedure has to be iterated N times, down to $\Lambda\,\mathrm{e}^{-Nl} \sim K_c$. The final equations are of the form (3.2) but apply to the flow restricted to wave numbers $0 < k < K_c$, consistent with the fact that we want to discuss the response of the system in the limit of small wavenumbers and frequencies. This however does not fully justify the fact that the small wavenumber and frequency limits are taken much earlier in the procedure. It is assumed that the intermediate equations actually remain valid on a much larger range of wavenumbers and frequencies.

As shown in the papers which inspired this calculation (Forster *et al.* 1977, and papers quoted in the introduction, it is convenient to approximate the recurrence equations obtained as differential equations in p, where $p \sim \Lambda\,\mathrm{e}^{-nl}$ is the wavenumber up to which the averaging procedure has been currently carried out. Rose (1977) computed corrections to the diffusion coefficient for passive scalar transport including near-grid effects: he shows that the result depends on the thickness of the shells and claims that thin shells probably give a more accurate result.

Introducing coefficients $C_n^{\perp}$, $N_n^{\perp}$, $M_n^{\perp}$ and $C_n^{\parallel}$, $N_n^{\parallel}$, $M_n^{\parallel}$ which are functions of $\nu_t p^2$, $\mu_t p^2$, $c_t p$, and Ω_J^2, with $n = -2, 0, 2 \ldots$ (see (B 16) to (B 21) in Appendix B), these differential equations are found to take the form:

$$-\frac{\mathrm{d}c_t^2}{\mathrm{d}p} = \frac{2}{3} I_0^{\perp}(p) + \sum_{j=\perp,\parallel,\, n=-2,0,\ldots} C_n^j(\nu_t p^2, \mu_t p^2, c_t p, \Omega_J^2)\, I_n^j(p), \tag{3.31}$$

$$-\frac{\mathrm{d}\nu_t}{\mathrm{d}p} = \frac{8}{15}\frac{1}{\nu_t p^2} I_0^{\perp}(p) + \sum_{j=\perp,\parallel,\, n=-2,0,\ldots} N_n^j(\nu_t p^2, \mu_t p^2, c_t p, \Omega_J^2)\, I_n^j(p), \tag{3.34}$$

$$-\frac{\mathrm{d}\mu_t}{\mathrm{d}p} = -\frac{4}{15}\frac{1}{\nu_t p^2} I_0^{\perp}(p) + \sum_{j=\perp,\parallel,\, n=-2,0,\ldots} M_n^j(\nu_t p^2, \mu_t p^2, c_t p, \Omega_J^2)\, I_n^j(p), \tag{3.35}$$

$$-\frac{\mathrm{d}\Omega_J^2}{\mathrm{d}p} = 0. \tag{3.36}$$

The minus sign in front of the derivatives comes from the fact that the renormalization proceeds towards smaller wavenumbers. Integration of these equations down to $p = K_c$ gives the renormalized coefficients ν_t, μ_t and c_t^2 that enter the perturbed equations of motion of the form (3.25), which apply to the flow restricted to wavenumbers $0 < k < K_c$. Note that the Jeans frequency is not renormalized. The consequences of the new equation are analysed below.

4. Analysis of stability

4.1. *Case with no gravity*

We first examine the behaviour of the *renormalized* propagator without gravity, where the system is expected to be stable. In this section we show that our result is consistent with the expectations. The renormalized c_t^2 and viscosity coefficients in this case are obtained from (3.33) and (3.34), when $\Omega_J = 0$.

4.1.1. *Turbulent pressure*

Equation (3.33) without expansion in ω_p can be re-expressed explicitly as

$$-\frac{\mathrm{d}c_t^2(p)}{\mathrm{d}p} = \tfrac{2}{3} I_0^{\perp}(p) + \tfrac{1}{3} I_0^{\parallel}(p) \left(\frac{2}{1 + \dfrac{\Omega_c^2 (c_t^2(p)\, p^2/\Omega_c^2 - 1)^2}{(\nu_t + \mu_t)^2(p)\, p^4}} - 1 \right). \tag{4.1}$$

In all cases the solenoidal component of the momentum density field increases c_t^2 by $\frac{2}{3} I_0^{\perp}$, very much like the thermal motions build up the thermal pressure. The potential component adds a correction of between $+\frac{1}{3} I_0^{\parallel}$ and $-\frac{1}{3} I_0^{\parallel}$. The possibility arises that c_t^2 decreases with decreasing wavenumbers, at small scales where the compressible component may be important. However, when c_t^2 approaches zero, the coefficient of $I_0^{\parallel}(p)$ becomes positive and prevents c_t^2 from getting negative. At this point we recall that we have assumed in §3.1 that compressibility effects dominate only at wavenumbers larger than the crossover K_x, so that, for wavenumbers smaller than K_x, c_t^2 will necessarily increase monotonically, and can be written in this scale domain as

$$c_t^2(p) = \frac{1}{3} \int_p^{K_x} \mathrm{d}r \left(2 I_0^{\perp}(r) - I_0^{\parallel}(r)\right) + c_t^2(K_x). \tag{4.2}$$

In the limit $p \to 0$ this integral is not simply $\frac{1}{3}\langle \phi^2 \rangle$.

4.1.2. *Turbulent shear viscosity* ν_t

The correction to the shear viscosity ν_t is written, in the case $c_t^2(p) > \Omega_c^2/p^2$

$$-\frac{d\nu_t(p)}{dp} = \frac{8}{15}\frac{I_0^{\perp}(p)}{\nu_t(p)\,p^2} + \frac{1}{3}\nu_t(p)\,p^2 I_{-2}^{\parallel}(p). \tag{4.3}$$

As shown above c_t^2 can never vanish, but may get very small. In the limit where $c_t^2 \to 0$ this equation is written

$$-\frac{d\nu_t(p)}{dp} = \left(\frac{8}{15}\frac{1}{\nu_t(p)\,p^2} + \frac{2}{5}\frac{1}{(\nu_t(p)+\mu_t(p))\,p^2}\right) I_0^{\perp}(p) - \frac{1}{3}\nu_t(p)\,p^2 I_{-2}^{\parallel}(p). \tag{4.4}$$

Note that even in the wavelength range where the compressible part of the velocity field dominates, any decrease of the viscosity due to the negative correction which appears in the case $c_t^2 \to 0$ will produce an increase of the positive term of the right-hand side of (4.4). As a consequence ν_t is bound to stay positive. For wavenumbers smaller than the crossover, the viscosity will increase monotonically, and can be written

$$\nu_t(p) = \left(\frac{16}{15}\int_p^{K_x} dr\,\frac{I_0^{\perp}(r)}{r^2}\exp\left[-\frac{2}{3}\int_p^r dq\,q^2 I_{-2}^{\parallel}(q)\right] + \nu_t^2(K_x)\exp\left[-\frac{2}{3}\int_p^{K_x} dq\,q^2 I_{-2}^{\parallel}(q)\right]\right)^{\frac{1}{2}}. \tag{4.5}$$

Note that the absolute value of the exponent in the above integrals is always less than

$$\frac{2}{3}\int_0^{\infty} dq\,q^2 I_{-2}^{\parallel}(q) \tag{4.6}$$

which is equal to $\frac{2}{3}$ of the relative density fluctuation $\langle n^2\rangle$ (see (3.18)). Under the assumptions of §3.1, this quantity is always much smaller than 1, and then the exponentials are of order 1.

4.1.3. *Coefficient* μ_t *and related turbulent bulk viscosity*

The coefficient μ_t of the $k_i k_j$ term is simply related to the viscosity coefficient for compressible modes, which is $\nu_t + \mu_t$, and to the bulk viscosity, equal to $\mu_t - \frac{1}{3}\nu_t$. The coefficient μ_t obeys the following differential equation:

$$-\frac{d\mu_t(p)}{dp} = -\frac{4}{15}\frac{I_0^{\perp}(p)}{\nu_t(p)\,p^2} + \frac{1}{3}\mu_t(p)\,p^2 I_{-2}^{\parallel}(p). \tag{4.7}$$

For wavenumbers smaller than the crossover, this coefficient will increase monotonically with decreasing p, and can be written

$$\mu_t(p) = \mu_t(K_x)\exp\left[\frac{2}{3}\int_p^{K_x} dq\,q^2 I_{-2}^{\parallel}(q)\right] - \frac{4}{15}\int_p^{K_x} dr\,\frac{I_0^{\perp}(r)}{\nu_t(r)\,r^2}\exp\left[\frac{1}{3}\int_p^r dq\,q^2 I_{-2}^{\parallel}(q)\right]. \tag{4.8}$$

A comparison of (4.5) and (4.8) shows that $|\mu_t|$ remains always smaller than ν_t. At small wavenumbers, where the terms in $I_0^{\perp}$ are dominating, it is easily seen from (4.3) and (4.7) that the coefficient μ_t tends to $-\frac{1}{2}\nu_t$. Although the bulk viscosity $\mu_t - \frac{1}{3}\nu_t$ is negative, the viscosity coefficient of the compressible modes, $\nu_t + \mu_t$, remains positive for any wavenumber. No instability can be generated by the viscous terms. No inverse cascade is expected.

4.2. *General case, Jeans stability*

We consider now the results including gravity. We point out that the Jeans frequency is not affected by the turbulence:

$$\mathrm{d}\Omega_{\mathrm{J}}^2/\mathrm{d}p = 0. \tag{4.9}$$

The corrections to c_{t}^2 and viscosity coefficients are slightly affected. In the case of positive $(c_{\mathrm{t}}^2 p^2 - \Omega_{\mathrm{J}}^2)$:

$$-\frac{\mathrm{d}c_{\mathrm{t}}^2(p)}{\mathrm{d}p} = \tfrac{2}{3}I_0^{\perp}(p) - \tfrac{1}{3}I_0^{\parallel}(p) - \tfrac{2}{3}(1+\mu_{\mathrm{t}}(p)/\nu_{\mathrm{t}}(p))\,\Omega_{\mathrm{J}}^2 I_{-2}^{\parallel}(p). \tag{4.10}$$

Instabilities can occur at large wavenumbers. As in §3.1 we consider wavenumbers below the crossover K_x, where the turbulence spectrum is dominated by the incompressible field. Below K_x the correction to c_{t}^2 is dominated by the first term on the right-hand side of (4.10). As in §3.1, c_{t}^2 keeps increasing, and the terms in $(\Omega_{\mathrm{J}}^2/(c_{\mathrm{t}}^2 p^2 - \Omega_{\mathrm{J}}^2)$ vanish. Integrating the remaining terms, and with the reasonable assumption of a constant value for $\mu_{\mathrm{t}}/\nu_{\mathrm{t}}$ (see §4.1.3), we get

$$c_{\mathrm{t}}^2(p) = c_{\mathrm{t}}^2(K_x) + \frac{1}{3}\int_p^{K_x} \mathrm{d}r \left(2I_0^{\perp}(r) - I_0^{\parallel}(r) - 2\left(1 = \frac{\mu_{\mathrm{t}}(r)}{\nu_{\mathrm{t}}(r)}\right)\Omega_{\mathrm{J}}^2 I_{-2}^{\parallel}(r)\right). \tag{4.11}$$

Let us now turn back to the linear analysis of the gravitational stability of large scales ($k < K_{\mathrm{c}}$) introduced in §2. We use the new equation of form (3.25) with the coefficients of the propagator replaced by the renormalized c_{t}^2 and renormalized viscosity coefficients computed for $p = K_{\mathrm{c}}$. The dispersion equation (3.4) now becomes

$$\omega_k^2 - c_{\mathrm{t}}^2(K_{\mathrm{c}})\,k^2 + \mathrm{i}\omega_k\,k^2(\nu_{\mathrm{t}}(K_{\mathrm{c}}) + \mu_{\mathrm{t}}(K_{\mathrm{c}})) + \Omega_{\mathrm{J}}^2 = 0. \tag{4.12}$$

This is formally the same dispersion equation as that obtained in the static case (3.4). Obviously turbulence acts as a stabilizing agent against self-gravity.

In order to be able to proceed analytically we had to assume that the turbulence was restricted to scales smaller than the scale k for which we wanted to assess the gravitational stability. However we conjecture that a similar result would hold at a scale k within the turbulent range. In this case the integration has to be stopped at k and the equation of dispersion is written:

$$\omega_k^2 - k^2 c_{\mathrm{t}}^2(k) + \mathrm{i}\omega_k\,k^2(\nu_{\mathrm{t}}(k) + \mu_{\mathrm{t}}(k)) + \Omega_{\mathrm{J}}^2 = 0. \tag{4.13}$$

This is just the equation discussed in our previous paper (Bonazzola *et al.* 1987), which had been justified there on the basis of two-dimensional numerical analysis. It is satisfactory to find here this same equation again as a result of a more deductive, though approximate, process.

We present in figure 2 comparison of the dispersion relations for Jeans waves from various treatments of the turbulence. Figure 2(*a*) shows the plain Jeans result, with non-renormalized sound speed, while figure 2(*b*) represents the solution of the same problem with renormalized pressure, but ignoring the effective viscosity of turbulence. Finally figure 2(*d*) shows the solution of (4.13), with all the effects included. As expected the main effect is the big widening of the stability range when the effect of turbulent pressure is taken into account. Figure 2(*c*) shows that the turbulent viscous effects, which should be included for consistency, alter this main

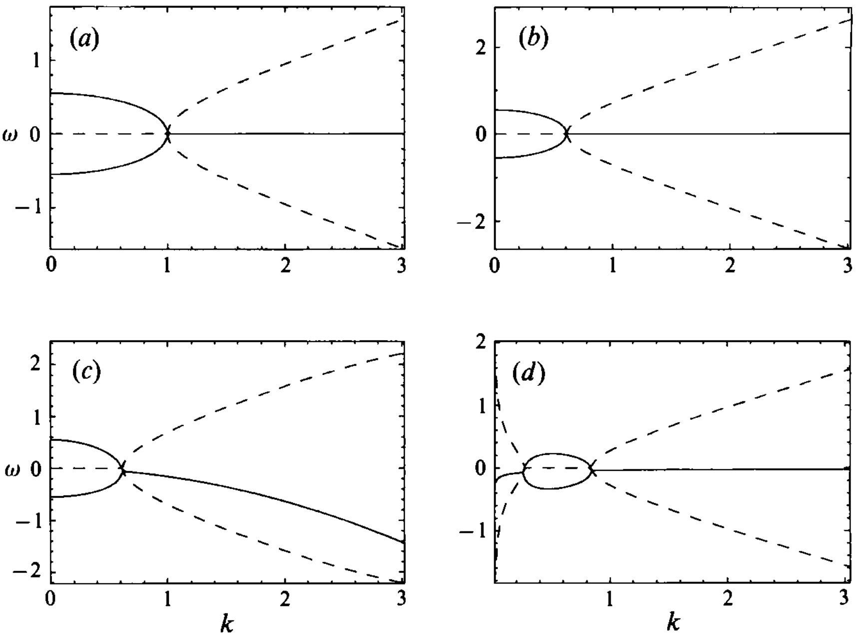


FIGURE 2. Dispersion relations for the Jeans problem. The dashed line represents the real part of the frequency and the solid line is the growth rate. They are normalized to the Jeans wavenumber and frequency. The abscissa on these plots is the wavenumber, normalized to the nonturbulent Jeans length. (*a*) Classical Jeans dispersion relation, i.e. with thermal pressure only included. (*b*) Dispersion relation including gas and turbulent pressure only. (*c*) Dispersion relation according to our (4.12), including gas and turbulent pressure, and both kinds of turbulent viscosity. (*d*) Dispersion relation according to the conjecture expressed by (4.13), i.e. with scale-dependent turbulent viscosity and turbulent pressure. The spectrum of the divergence-free component of the velocity is assumed to be a power law $\propto k^{-\alpha}$ with $\alpha = 4$, larger than the critical value $\alpha = 3$ (cf. Bonazzola *et al.* 1987). Two regimes of stability appear.

effect only in quantitative aspects, near the threshold for example, and add extra stability.

5. Conclusion

Our goal here has been only to study the effect of compressible turbulence on gravitational stability at large scale, not to build a model of compressible turbulence. We have studied the linear stability of a self-gravitating turbulent fluid using a renormalization technique. We have shown that, under the assumptions on the turbulent spectra made in §3.1, the effect of the response of the small-scale turbulence to the large-scale perturbation (in the limit $k \to 0, \omega \to 0$) can be described in a new equation applicable only to large scales, by a renormalized c_t^2 and renormalized viscosity coefficients ν_t and μ_t, which replace the ordinary quantities of the general equation. We have also shown how the effective viscosity itself contributes to the renormalization of the pressure. For spatial and temporal scales much larger than those of the turbulent spectrum considered, our result shows that $\rho c_t^2 = \rho\, \partial P_t/\partial \rho$ is equal to $\frac{2}{3}$ of the energy density of the transverse motions, in complete analogy with the kinetic pressure. The longitudinal motions contribute at most $\frac{2}{3}$ of their energy density. In the plausible case of turbulence dominated at large scales by the incompressible flow, the turbulent pressure is essentially provided by this flow.

Under the assumptions adopted, we have shown that the Jeans stability problem, obeys, at large scales, a dispersion relation which includes renormalized pressure and viscosities. Our results justify the introduction of turbulent pressure in the discussion of gravitational stability and exhibit its stabilizing role against gravitational collapse. As explained at the end of §4, an extension of these results to scales within the turbulent range will make these quantities scale-dependent, and lead to the dispersion relation proposed in our earlier paper (Bonazzola *et al.* 1987), which receives here some more basic justification. The consequences of this 'renormalized' dispersion relation, which are numerically illustrated in our figure 2, confirm the conclusions of the above quoted paper concerning the stability properties of turbulent clouds.

Some further aspects which might deserve some attention are related to characteristic timescales. Our calculations assume that the free-fall time, as well as the typical timescale of the perturbation considered are both very large when compared to the turbulent characteristic timescales. Preliminary results of new numerical simulations indicate that the turbulent pressure requires a minimum time to be established (this is the dynamical timescale of the turbulence). It should also act for a time that should be limited, because the turbulent flow would leak, by diffusion, out of the region of fluid where it is present at time t. A consideration of such effects, which bear on the turbulent energy content, would require the solution of the full nonlinear problem.

To discuss the collapse problem in terms of the development of a linear perturbation on a state unperturbed at large scales by turbulence, we had to assume a complete separation of spatial scales between the turbulence spectrum and the collapsing unit. This limitation is likely to be removable but, then, the 'linear Jeans stability' formulation will not be adequate any more, because the unstable motions at the scale of the collapsing unit will interfere with the bulk of the turbulent motions and a linear stability analysis will not make sense any more. The scope will thus have to be radically changed to the consideration of self-gravitating, compressible, developed turbulence. Though this is quite obvious, we think that our approach has made this point very clear. As stated in the introduction in this case the problem involves other aspects, namely the role of timescales mentioned above, the generation (Sasao 1973) and disruption (Léorat *et al.* 1990) of density enhancements by the shearing flow itself.

Other aspects of compressible turbulence have been treated in the literature (Higdon 1986; Henriksen 1986; Hartke, Canuto & Alonso 1988), but a complete treatment of gravitational instability in the presence of turbulence is still lacking.

It is a great pleasure to thank A. Pouquet for encouraging us to persevere in this study, H. K. Moffatt for sending a copy of his work on the RNG technique applied to hydrodynamics, and S. Fauve, R. Henriksen and D. Hollenbach for comments on early versions of the manuscript.

Appendix A. Remarks on the papers by Chandrasekhar (1951*a*, *b*)

Chandrasekhar (1951*a*, *b*) obtained an equation for the time evolution of the density correlation which includes a turbulent pressure term. Sasao (1973) notes that Chandrasekhar did not consistently use the hypothesis of joint normal distributions, and ignored terms among which is Sasao's term for the generation of density fluctuations. We think that there is an additional point to make.

In his equation (19), Chandrasekhar (1951*b*) also neglected additional terms (in the limit $r \to \infty$), and we feel that he did not do that consistently: he dropped all terms involving velocity correlations, but kept a particular term which adds a turbulent pressure to the thermal pressure. The resulting equation for the density correlation is then his equation (20). In fact, the neglected terms are larger than this single term because

$$v^2 \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial}{\partial r}\langle \delta\rho\, \delta\rho' \rangle\right) \sim \langle v^2 \rangle \frac{1}{r^2} \langle \delta\rho^2 \rangle C_{\rho\rho}(r), \tag{A 1}$$

$$-2 \frac{\partial^2}{\partial \xi_i \partial \xi_j} \langle \rho\rho' \rangle \langle v_i v_j' \rangle \sim \frac{1}{r^2} \langle \rho^2 \rangle \langle v^2 \rangle C_{v_i v_j}(r), \tag{A 2}$$

where $C_{\rho\rho}(r)$ and $C_{v_i v_j}(r)$ are the correlation functions of the density fluctuation $\delta\rho$ and velocity field v respectively, as a function of distance r, ξ_i being the components of r. These correlation functions are related through the linearized continuity equation, in Fourier space, for pulsation ω, and wavenumber k:

$$\omega \delta\hat{\rho}(\boldsymbol{k}) \sim \langle \rho \rangle \boldsymbol{k} \cdot \hat{\boldsymbol{v}}(\boldsymbol{k})$$

so that
$$\langle \delta\hat{\rho}(\boldsymbol{k})\, \delta\hat{\rho}^*(\boldsymbol{k}) \rangle \sim k^2/\omega^2 \langle \rho^2 \rangle \langle \hat{v}^{\parallel}(\boldsymbol{k})\, \hat{v}^{*\parallel}(\boldsymbol{k}) \rangle,$$

where $v^{\parallel}$ is the compressible part of the velocity field. If $k^2/\omega^2 = c_s^2$ (sound waves), and using the fact that

$$\langle \hat{v}^{\parallel}(\boldsymbol{k})\, \hat{v}^{*\parallel}(\boldsymbol{k}) \rangle \sim \langle v^2 \rangle / c_s^2 \langle \hat{v}(\boldsymbol{k})\, \hat{v}^*(\boldsymbol{k}) \rangle$$

we deduce after inverse Fourier transformation

$$\frac{\langle \delta\rho^2 \rangle}{\langle \rho \rangle^2} C_{\rho\rho}(r) \sim \frac{\langle v^2 \rangle^2}{c_s^2} C_{vv}(r).$$

This implies that equation (A 1) is approximately equal to (A 2) multiplied by $\langle v^2 \rangle^2 / c_s^2$, meaning that in the subsonic case, considered by Chandrasekhar, the neglected term is larger than the one retained.

To the same approximation level terms like $\langle \delta\rho\, \delta\rho' \rangle \langle v_i v_j' \rangle$ and $\langle \rho v_i' \rangle \langle \rho v_j' \rangle = \langle \delta\rho\, v_i' \rangle \langle \delta\rho\, u_j' \rangle$ should be neglected, and the asymptotic equation for the density correlation (equation (20) of Chandrasekhar 1951*b*) should be read

$$\frac{\partial^2 \langle \delta\rho\, \delta\rho' \rangle}{\partial t^2} = 2c_s^2 \Delta \langle \delta\rho\, \delta\rho' \rangle + 8\pi G \langle \rho \rangle \langle \delta\rho\, \delta\rho' \rangle - 2 \frac{\partial}{\partial \xi_i} \frac{\partial}{\partial \xi_j} \langle \rho \rangle^2 \langle v_i^{\perp} v_j^{\perp} \rangle,$$

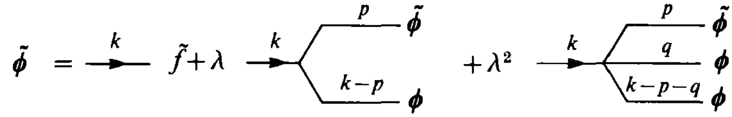
where $v^{\perp}$ is the solenoidal part of the velocity field. This equation is quite different from the equation discussed by Chandrasekhar.

In Chandrasekhar (1951*a*) the viscosity terms are discarded in a similar way (Chandrasekhar's (34)) leading to a wave equation for the density correlation without viscous damping, regardless of the value of the viscosity. This is, we believe, a non-physical feature of the mathematical development.

Appendix B. Renormalization of the Jeans problem with turbulence

B.1. *Description of the renormalization procedure*

The starting equation is (3.25), which we rewrite here with explicit mention of the wavenumbers:

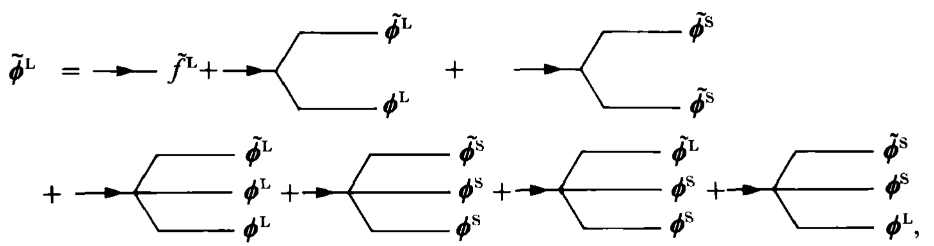


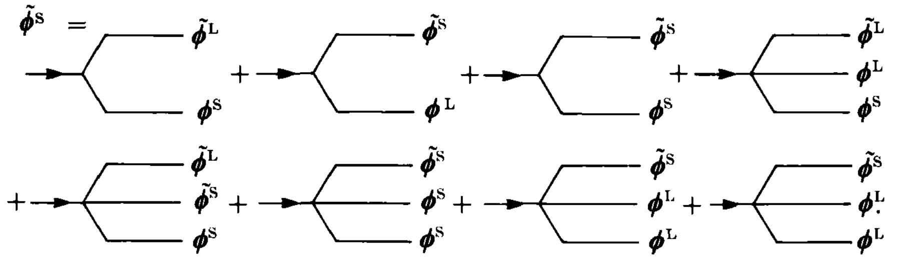
(B 1)

In this equation the perturbed quantities are represented with a tilda; the arrow represents the propagator, the vertices represent the nonlinear couplings of second and third order respectively. In Fourier space the vertices give convolutions over wavenumber and frequency. The expressions for the kernels are given in (3.21)–(3.23).

As explained in §2 this expression represents the linear perturbation development of (3.14) in Fourier space. Equation (3.14) was obtained from the conservation equations (3.1)–(3.3), by dropping nonlinear couplings of order strictly larger than 3; terms of order 3 are kept because they give contributions of the same order of magnitude as terms of order 2 in the perturbation development at lowest order.

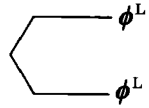
We now proceed with the separation in two scales as explained in §2. The equations for low ($k \leqslant \Lambda \mathrm{e}^{-l}$) and high ($\Lambda \mathrm{e}^{-l} < k \leqslant \Lambda$) wavenumbers, respectively represented by L and S superscripts, are


(B 2)


(B 3)

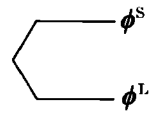
The integrals corresponding to the vertices have been split on different integration domains, $p \leqslant \Lambda \mathrm{e}^{-l}$ and $\Lambda \mathrm{e}^{-l} < p \leqslant \Lambda$. As the expressions for the vertices given in (3.22) and (3.23) are not symmetrical, care has to be taken in the choice of the domain of integration. Note that this problem does not appear in the incompressible hydrodynamics considered by Forster *et al.* (1977), where the corresponding vertex is symmetric. A simple way to deal with this problem is to symmetrize the expressions of the vertices with respect to the outgoing branches.

In writing (B 2) and (B 3) we dropped the near-grid coupling, for example terms like



(B 4)

in the high-wavenumber equation, or



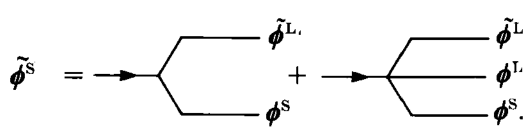
(B 5)

in the low-wavenumber equation. A correct treatment of the effect of near-grid coupling would involve a numerical treatment of averaging integrals (Rose 1977), which we do not feel worthwhile in the present context. It is usually conjectured that the essence of the physics of the eddy interaction remains at least quantitatively preserved despite this.

A second bold approximation consists in simplifying the integration domains in the computation of vertex integrals. For example the convolution of quantities of high wavenumbers p and $k-p$ should be made with: $\Lambda\,\mathrm{e}^{-l} < p \leqslant \Lambda$ and $\Lambda\,\mathrm{e}^{-l} < |k-p| \leqslant \Lambda$. But only the condition $\Lambda\,\mathrm{e}^{-l} < p \leqslant \Lambda$ is kept. This condition is obtained by Forster *et al.* (1977) by a change of variable $p' = p-\frac{1}{2}k$, with $\Lambda\,\mathrm{e}^{-l} < p' \leqslant \Lambda$. But this is correct only in the limit $k \rightarrow 0$. It is not obvious at all that it is permissible to take this limit before making the computations.

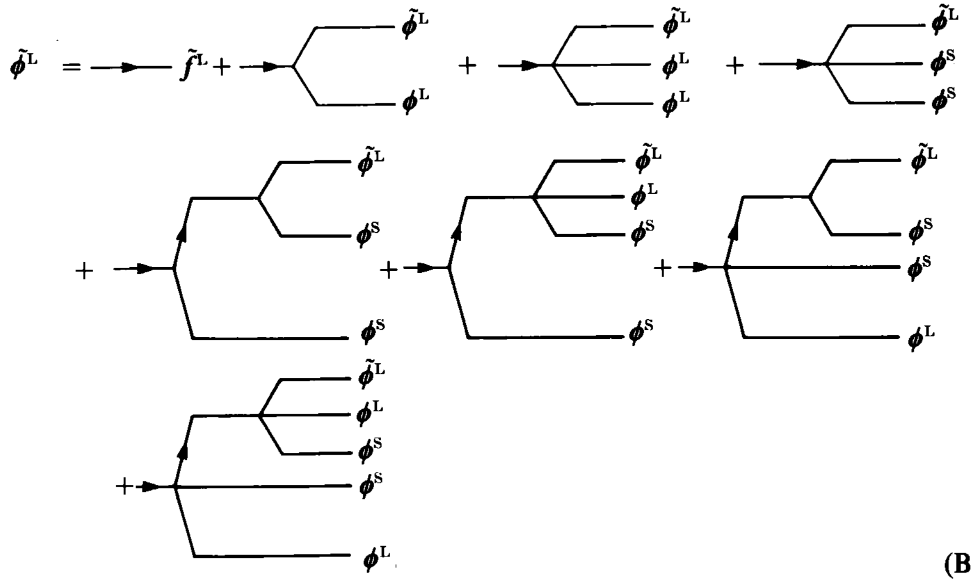
The presentation adopted by Moffatt (1983) consits in assuming a discrete spectrum concentrated around wavenumbers $k_1, k_2, \ldots, k_n$ with $k_1 \gg k_2 \gg \ldots k_n$. This avoids the difficulties related to the near-grid coupling and the change of variables. But in the continuous case, representing the spectrum with such a discrete sampling is another bold approximation.

We obtain a lowest-order solution of (B 3) by keeping only the first two terms, as the other ones involve the perturbation at large wavenumber, or include nonlinearities at large wavenumber:



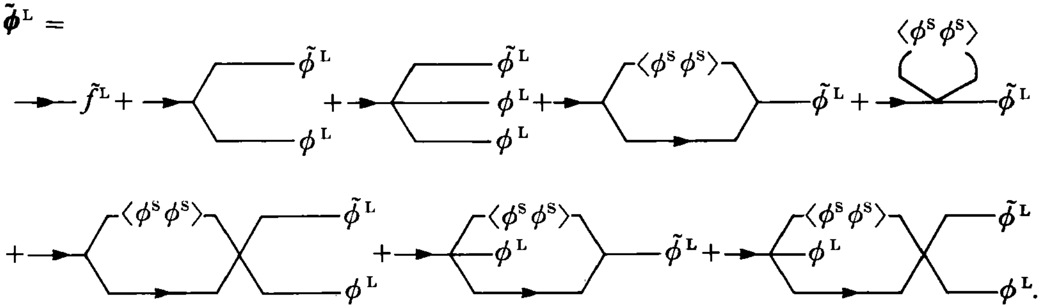
(B 6)

Replacing the perturbed momentum density at high wavenumber in the equation for low wavenumber by this solution gives, at order λ^2 in nonlinear couplings,



(B 7)

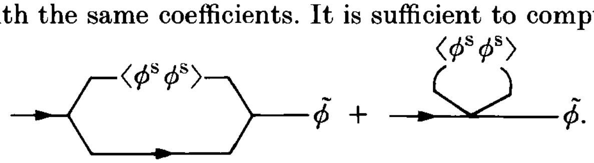
Averaging for higher wavenumbers we are left with the following terms:



(B 8)

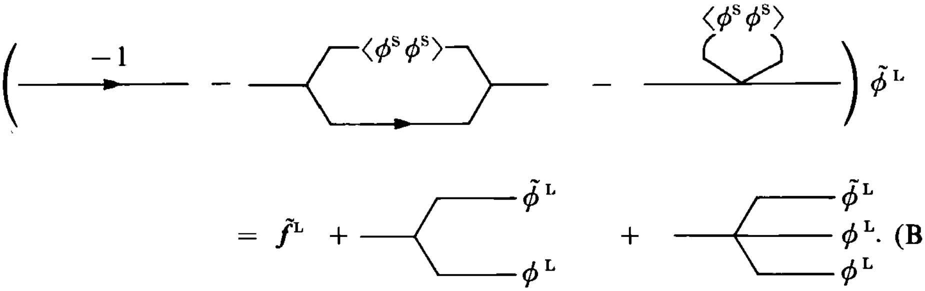
This is a first-order smoothing approximation, which consists in dropping terms like $\phi\phi - \langle\phi\phi\rangle$. This approximation is adequate if the effective Reynolds number for the large wavenumber is small.

Using the formal development of §2 we know that the graphs correcting the vertices simply give corrections to nonlinear terms coming from the development of $1/\rho$ with the same coefficients. It is sufficient to compute the linear corrections:



(B 9)

The double ϕ^{S} represents the correlation function of momentum (3.12) for high wavenumbers. Multiplied by the inverse of the propagator (represented by an arrowed line, with -1 on top of it) the low-wavenumber equation can be written:



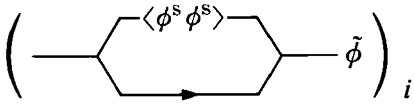
(B 10)

The two surviving graphs appear to be subtractive corrections to the inverse propagator. This new equation has the same structure as the original equation (3.25). The two differences are a wavenumber range reduced to $0 \leqslant k \leqslant \Lambda\,\mathrm{e}^{-l}$ and a modified propagator. The corrections to the propagator are computed in §B.2. The pressure and viscosity are replaced by turbulent coefficients including the effects of turbulence at wavenumber higher than $\Lambda\,\mathrm{e}^{-l}$.

In the cases where the corrections increase the viscosity sufficiently, the effective Reynolds number at $k = \Lambda\,\mathrm{e}^{-l}$ is decreased to a value sufficiently low so that the whole procedure may be repeated for wavenumbers $0 \leqslant k \leqslant \Lambda\,\mathrm{e}^{-l}$, with a new separation between small $(0 \leqslant k \leqslant \Lambda\,\mathrm{e}^{-2l})$ and large $(\Lambda\,\mathrm{e}^{-2l} < k \leqslant \Lambda\,\mathrm{e}^{-l})$ wavenumbers. The operation is iterated until all turbulent wavenumbers down to K_{c} have been swallowed in the progressive average.

B.2. *Computation of the lowest-order graphs*

The graph:

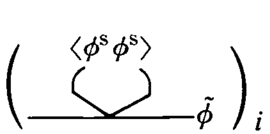


(B 11)

must be read as

$$\iint \mathrm{d}\boldsymbol{p}\,\mathrm{d}\boldsymbol{q}\,\mathscr{Q}_{ilm}(\boldsymbol{k},\boldsymbol{p},\boldsymbol{k}-\boldsymbol{p})\,\mathscr{G}_{mn}(\boldsymbol{k}-\boldsymbol{p})\,\mathscr{Q}_{nrj}(\boldsymbol{k}-\boldsymbol{p},\boldsymbol{q},\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q})\,\langle\phi_l(\boldsymbol{p})\,\phi_r(\boldsymbol{q})\rangle\,\tilde{\phi}_j(\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q}). \tag{B 12}$$

And the graph



(B 13)

must be read as

$$\iint \mathrm{d}\boldsymbol{p}\,\mathrm{d}\boldsymbol{q}\,\mathscr{S}_{ilmj}(\boldsymbol{k},\boldsymbol{p},\boldsymbol{q},\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q})\,\langle\phi_l(\boldsymbol{p})\,\phi_m(\boldsymbol{q})\rangle\,\tilde{\phi}_j(\boldsymbol{k}-\boldsymbol{p}-\boldsymbol{q}). \tag{B 14}$$

The propagator is the inverse of (3.21) and the vertices are obtained by symmetrizing (3.22) and (3.23) with respect to the outgoing branches.

These integrals can be computed analytically in the limit of small k and ω_k under the following assumptions. The integrand is developed up to order k^2. The integral over angles is performed under the assumption of isotropy. Knowledge of the frequency dependence of $\psi^{\perp}$ and $\psi^{\parallel}$ is necessary to perform the frequency integration. We chose a spectrum strongly peaked at frequency Ω_c, allowing expansion in powers of ω_p.

All correction terms obtained under these assumptions may be grouped as factors of $\mathrm{i}/\omega_k k_i k_j$, $k^2\delta_{ij}$ or $k_i k_j$. They are interpreted as modifying either the pressure c_s^2 or the viscosity coefficients ν_0 and μ_0, which are factors of the same quantities in the expression of the inverse propagator (3.21). There are no factors of $(\mathrm{i}/\omega_k)(k^2)/k_i k_j)$, and thus no correction to the Jeans frequency. The correction to the inverse propagator reads

$$\Delta\mathscr{G}_{ij}^{-1}(\boldsymbol{k}) = \Delta c_t^2 \frac{\mathrm{i}}{\omega_k} k_i k_j + \Delta\nu_t k^2\delta_{ij} + \Delta\mu_t k_i k_j. \tag{B 15}$$

The corrections are expressed as functions of frequency momenta of the spectrum, $I_n^{\perp}(p)$ and $I_n^{\parallel}(p)$, defined in the main text. For each of the three corrections, the two highest-order non-vanishing terms for each of $I_n^{\perp}(p)$, $I_n^{\parallel}(p)$, and $\Omega_J^2 I_n^{\parallel}(p)$ contributions are kept. It is recalled that $I_{-2}^{\parallel}(p)$ is proportional to the power spectrum of density fluctuations: it remains finite.

In the limit where Ω_c is much smaller than $\nu_t p^2$, $\mu_t p^2$ and $c_t p$, we obtain for each of the three individual corrections

$$\begin{aligned}\Delta c_t^2 = {} & \frac{2}{3} I_0^{\perp}(p) - \frac{4}{3}\frac{1}{\nu_t^2 p^4} I_2^{\perp}(p) - \tfrac{1}{3} I_0^{\parallel}(p) + \frac{2}{3}\frac{(\nu_t+\mu_t)^2 p^4}{(c_t^2p^2-\Omega_J^2)^2} I_2^{\parallel}(p) - \frac{2}{3}\left(1+\frac{\mu_t}{\nu_t}\right)\Omega_J^2 I_{-2}^{\parallel}(p) \\ & + \left(\frac{2}{5}\frac{1}{c_t^2p^2-\Omega_J^2} - \frac{(\nu_t+\mu_t)^2p^4}{(c_t^2p^2-\Omega_J^2)^2} + \frac{2}{15}\frac{17\nu_t+5\mu_t}{\nu_t^3p^4}\right)\Omega_J^2 I_0^{\parallel}(p),\end{aligned} \tag{B 16}$$

$$\begin{aligned}\Delta\nu_t = {} & \frac{8}{15}\frac{1}{\nu_t p^2} I_0^{\perp}(p) + \left(-\frac{16}{15}\frac{1}{\nu_t^3p^6} + \frac{2}{5}\frac{(\nu_t+\mu_t)p^2}{(c_t^2p^2-\Omega_J^2)^2}\right) I_2^{\perp}(p) \\ & + \frac{1}{3}\nu_t p^2 I_{-2}^{\parallel}(p) + \left(\frac{1}{15}\frac{(21\nu_t+\mu_t)p^2}{c_t^2p^2-\Omega_J^2} - \frac{2}{3}\frac{(\nu_t+\mu_t)^2\nu_t p^6}{(c_t^2p^2-\Omega_J^2)^2}\right) I_0^{\parallel}(p) \\ & + \frac{1}{15}\frac{(\nu_t+\mu_t)p^2}{c_t^2p^2-\Omega_J^2}\Omega_J^2 I_{-2}^{\parallel}(p) - \left(\frac{3}{5}\frac{(\nu_t+\mu_t)p^2}{(c_t^2p^2-\Omega_J^2)^2} + \frac{1}{15}\frac{(\nu_t+\mu_t)^3p^6}{(c_t^2p^2-\Omega_J^2)^3}\right)\Omega_J^2 I_0^{\parallel}(p),\end{aligned} \tag{B 17}$$

$$\begin{aligned}\Delta\mu_t = {} & \frac{-4}{15}\frac{1}{\nu_t p^2} I_0^{\perp}(p) + \left(\frac{68}{15}\frac{1}{\nu_t^3p^6} + \frac{2}{15}\frac{(\nu_t+\mu_t)p^2}{(c_t^2p^2-\Omega_J^2)^2}\right) I_2^{\perp}(p) + \tfrac{1}{3}\mu_t p^2 I_{-2}^{\parallel}(p) \\ & + \left(\frac{1}{15}\frac{(17\nu_t+37\mu_t)p^2}{c_t^2p^2-\Omega_J^2} - \frac{2}{3}\frac{(\nu_t+\mu_t)^2\mu_t p^6}{(c_t^2p^2-\Omega_J^2)^2}\right) I_0^{\parallel}(p) \\ & + \left(\frac{2}{3}\frac{\nu_t+\mu_t}{\nu_t^2p^2} - \frac{8}{15}\frac{(\nu_t+\mu_t)p^2}{c_t^2p^2-\Omega_J^2}\right)\Omega_J^2 I_{-2}^{\parallel}(p) \\ & + \left(-\frac{2}{15}\frac{47\nu_t+15\mu_t}{\nu_t^4p^6} - 4\frac{(\nu_t+\mu_t)p^2}{(c_t^2p^2-\Omega_J^2)^2} + \frac{38}{15}\frac{(\nu_t+\mu_t)^3p^6}{(c_t^2p^2-\Omega_J^2)^3}\right)\Omega_J^2 I_0^{\parallel}(p).\end{aligned} \tag{B 18}$$

In the case where c_t^2 would go down to very low values (this might happen if compressible motions are dominant at small scales), the above expansion is not valid.

Therefore, we also compute the corrections in the limit where Ω_c is much smaller than $\nu_t p^2$, $\mu_t p^2$ but $c_t p$ is much smaller than Ω_c. We find instead

$$\Delta c_t^2 = \frac{2}{3} I_0^{\perp}(p) - \frac{4}{3}\frac{1}{\nu_t^2 p^4} I_2^{\perp}(p) + \frac{1}{3} I_0^{\parallel}(p) - \frac{2}{3}\frac{1}{(\nu_t+\mu_t)^2 p^4} I_2^{\parallel}(p)$$

$$-\frac{1}{3}\left(5+\frac{2\mu_t}{\nu_t}\right)\Omega_J^2\ I_{-2}^{\parallel}(p) + \left(\frac{2}{15}\frac{17\nu_t+5\mu_t}{\nu_t^3 p^4} + \frac{7}{5}\frac{1}{(\nu_t+\mu_t)^2 p^4}\right)\Omega_J^2 I_0^{\parallel}(p), \tag{B 19}$$

$$\Delta\nu_t = \left(\frac{8}{15}\frac{1}{\nu_t p^2} + \frac{2}{5}\frac{1}{(\nu_t+\mu_t)p^2}\right) I_0^{\perp}(p) + \left(-\frac{16}{15}\frac{1}{\nu_t^3 p^6} - \frac{2}{5}\frac{1}{(\nu_t+\mu_t)^3 p^6}\right) I_2^{\perp}(p)$$

$$-\frac{1}{3}\nu_t p^2 I_{-2}^{\parallel}(p) - \frac{1}{15}\frac{9\nu_t-\mu_t}{(\nu_t+\mu_t)^2 p^2} I_0^{\parallel}(p)$$

$$-\frac{4}{15}\frac{(4\nu_t-\mu_t)^2}{(\nu_t+\mu_t)p^2}\Omega_J^2 I_{-2}^{\parallel}(p) + \frac{4}{15}\frac{8\nu_t+3\mu_t}{(\nu_t+\mu_t)^4 p^6}\Omega_J^2 I_0^{\parallel}(p), \tag{B 20}$$

$$\Delta\mu_t = \left(\frac{-4}{15}\frac{1}{\nu_t p^2} + \frac{2}{15}\frac{1}{(\nu_t+\mu_t)p^2}\right) I_0^{\perp}(p) + \left(\frac{68}{15}\frac{1}{\nu_t^3 p^6} - \frac{2}{15}\frac{1}{(\nu_t+\mu_t)^3 p^6}\right) I_2^{\perp}(p)$$

$$-\frac{1}{3}\mu_t p^2 I_{-2}^{\parallel}(p) - \frac{1}{15}\frac{3\nu_t+13\mu_t}{(\nu_t+\mu_t)^2 p^2} I_0^{\parallel}(p) + \left(\frac{2}{3}\frac{\nu_t+\mu_t}{\nu_t^2 p^2} + \frac{1}{15}\frac{13\nu_t-7\mu_t}{(\nu_t+\mu_t)^2 p^2}\right)\Omega_J^2 I_{-2}^{\parallel}(p)$$

$$+\left(-\frac{1}{15}\frac{67\nu_t+47\nu_t}{(\nu_t+\mu_t)^4 p^6} - \frac{2}{15}\frac{47\nu_t+15\mu_t}{\nu_t^4 p^6}\right)\Omega_J^2 I_0^{\parallel}(p). \tag{B 21}$$

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